The semigroup of star partial homeomorphisms of a finite deminsional Euclidean space

Oleg Gutik
(Ivan Franko National University of Lviv, Universytetska Str., 1, 79000, Lviv, Ukraine)
E-mail: `oleg.gutik@lnu.edu.ua`

Kateryna Melnyk
(Ivan Franko National University of Lviv, Universytetska Str., 1, 79000, Lviv, Ukraine)
E-mail: `chepil.kate@gmail.com`

We follow the terminology of [1, 2].

We shall introduce the notion of a star partial homeomorphism of a finite dimensional Euclidean space $\mathbb{R}^n$. By $\mathbf{St}_0(\mathbb{R}^n)$ we denote the set of all stars at the origin $\mathbf{0}$ of $\mathbb{R}^n$.

We describe the structure of the semigroup $\mathbf{PStH}_{\mathbb{R}^n}$ of star partial homeomorphisms of the space $\mathbb{R}^n$.

Proposition 1. *$\mathbf{PStH}_{\mathbb{R}^n}$ is an inverse submonoid of the symmetric inverse monoid $\mathscr{I}_{\mathfrak{c}}$.*

Proposition 2. (*i*) *An element α of $\mathbf{PStH}_{\mathbb{R}^n}$ is an idempotent if and only if $\alpha\colon S \to S$ is the identity map for some star $S \in \boldsymbol{St}_0(\mathbb{R}^n)$.*

(*ii*) *The band $E(\mathbf{PStH}_{\mathbb{R}^n})$ is isomorphic to the semilattice $(\boldsymbol{St}_0(\mathbb{R}^n), \cap)$.*

(*iii*) *$\varepsilon \leqslant \iota$ in $E(\mathbf{PStH}_{\mathbb{R}^n})$ if and only if $\operatorname{dom} \varepsilon \subseteq \operatorname{dom} \iota$.*

(*iv*) *$\alpha \leqslant \beta$ in $\mathbf{PStH}_{\mathbb{R}^n}$ if and only if $\beta|_{\operatorname{dom} \alpha} = \alpha$.*

Proposition 3. *Let be $\alpha, \beta \in \mathbf{PStH}_{\mathbb{R}^n}$. Then the following statements hold:*

(*i*) *$\alpha \mathscr{R} \beta$ in $\mathbf{PStH}_{\mathbb{R}^n}$ if and only if $\operatorname{ran} \alpha = \operatorname{ran} \beta$;*

(*ii*) *$\alpha \mathscr{L} \beta$ in $\mathbf{PStH}_{\mathbb{R}^n}$ if and only if $\operatorname{dom} \alpha = \operatorname{dom} \beta$;*

(*iii*) *$\alpha \mathscr{H} \beta$ in $\mathbf{PStH}_{\mathbb{R}^n}$ if and only if $\operatorname{ran} \alpha = \operatorname{ran} \beta$ and $\operatorname{dom} \alpha = \operatorname{dom} \beta$.*

Proposition 4. *$\mathbf{PStH}_{\mathbb{R}^n}$ is a bisimple inverse semigroup.*

Corollary 5. *Every two maximal subgroup in $\mathbf{PStH}_{\mathbb{R}^n}$ are isomorphic. Moreover every maximal subgroup in $\mathbf{PStH}_{\mathbb{R}^n}$ id isomorphic to the group of all star homeomorphisms of the unit ball $\mathbf{B}_1$ in $\mathbb{R}^n$.*

Theorem 6. *Every non-unit congruence on $\mathbf{PStH}_{\mathbb{R}^n}$ is a group congruence.*

References

[1] Mark V. Lawson. *Inverse Semigroups. The Theory of Partial Symmetries*, Singapore: World Scientific, 1998.

[2] Maria Moszyńska, *Selected Topics in Convex Geometry*, Basel: Birkhäuser, 2005.